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PLANCK FUNCTIONS AND INTEGRALS; METHODS OF COMPUTATION

by Thomas E. Michels

Goddard Space Flight Center
Greenbelt, Md.

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ABSTRACT

Black-body radiation is investigated with emphasis placed on a solution of the Planck integral,

$$\int_{\nu}^{\infty} B_{\nu,T}\, d\nu \ ,$$

suitable for computer and hand calculation. A computer program using this method has been included as well as graphs and tables of Planck functions and Planck integrals, which can be used for a wide range of wave numbers, ν, $(= \lambda^{-1})$, and temperatures, T, in degrees Kelvin. A suitable range is: ν from 0.001 to 300,000 cm^{-1} (λ from 10 meters to 333.3 angstroms) for temperatures between 1°K and 25,000°K.

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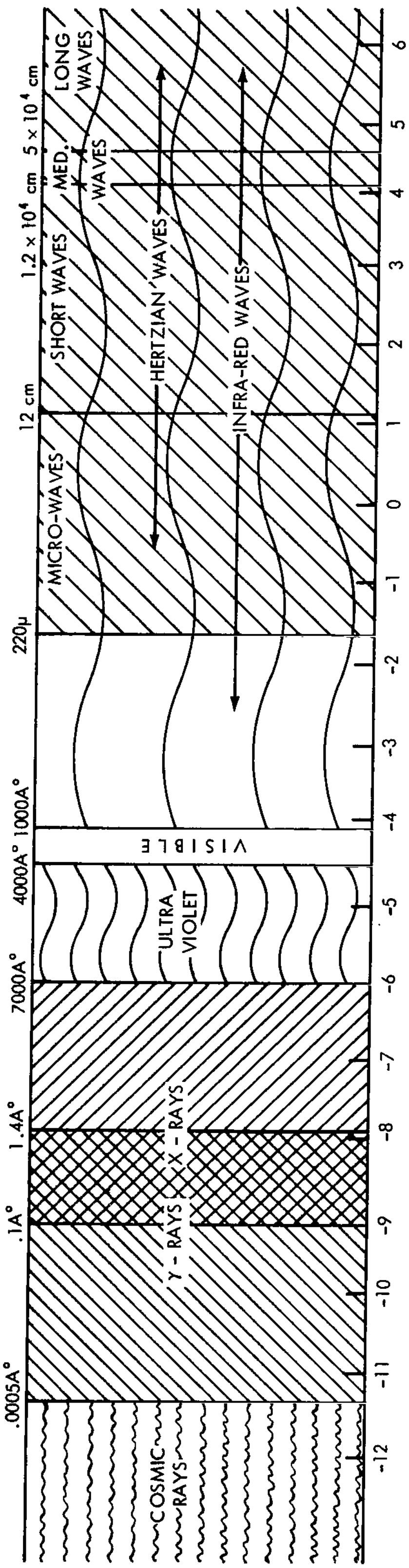

THE ELECTROMAGNETIC SPECTRUM

PLANCK FUNCTIONS AND INTEGRALS; METHODS OF COMPUTATION

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Thomas E. Michels

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INTRODUCTION

In practically any area of physics related to radiation pyrometry, spectroscopy, radiative transfer, or any other area of study of electromagnetic radiation, a need for the knowledge of the black-body spectrum is needed.

The visible windows seen on the surface of the earth in earth-based astronomical spectroscopy are of primary importance; in radiometry or infrared spectroscopy, only a small region of the electromagnetic spectrum, say from 5μ to 30μ, is of present concern. With the advances in filtering technology many new instruments have been developed that have necessitated the determination of the total energy or radiation intensity seen through these short radiation windows. In addition, with the advances of computer technology it has been realized that simulations of many laboratory as well as theoretical experiments can be accomplished on the computer; this, of course, leads to a need for computer calculation of the total radiation seen through these short windows. Therefore, it seemed desirable to have access to a short concise reference on black-body radiation (including useful tables of related Planck functions) and a numerical solution for computer and hand calculation of the total intensity seen over a specified range of the spectrum.

With these thoughts in mind, it was realized that it would be beneficial to have available a short concise reference on black-body radiation, including useful tables of related Planck functions and, as well, a numerical solution for computer and hand calculation of the total intensity seen over a specified range of the spectrum.

In order to compute the total black-body intensity seen through a window of say λ_1 to λ_2, we compute the integral from λ_1 to λ_2 of the Planck function $B_{\lambda,T}$; that is

$$\int_{\lambda_1}^{\lambda_2} B_{\lambda,T}\, d\lambda = \int_0^{\lambda_2} B_{\lambda,T}\, d\lambda - \int_0^{\lambda_1} B_{\lambda,T}\, d\lambda ,$$

where $B_{\lambda,T}$ represents the power radiated per unit wavelength interval at wavelength, λ, by a unit area of a black body at temperature, T, in degrees Kelvin. And, as will be seen, very convenient graphs of

$$\int_0^{\lambda} B_{\lambda,T}\, d\lambda$$

can be made of the integrated intensity for practically any range of the electromagnetic spectrum. This can also be done with the Planck function.

Since most areas of study now use wave number notation, ν (= λ^{-1}), all functions and tables have been presented using ν, although comparisons with wavelength have been included, and all functions have been measured in hemispherical radiation, implying that the units of the total intensity are watts per square centimeter per steradian. The tables of the Planck Function, Planck Integral, and related functions presented in Appendix C can be used for practical ranges of ν = 0.001 cm^{-1} to 300,000 cm^{-1} (or $\lambda = 0.333 \cdot 10^{-5}$ cm 1000 cm) and for temperatures ranging from 1°K to 25,000°K. Graphs are included that cover these spectrum ranges of sources over a wide temperature range.

A solution of the integral

$$\int_0^{\lambda} B_{\lambda,T}\, d\lambda = \int_{\nu}^{\infty} \frac{B_{\nu,T}}{\nu^2}\, d\nu$$

is presented for computer and hand calculation, and a listing of a function subprogram employing this method is presented in Appendix D.

BLACK-BODY RADIATION LAW

It has been found experimentally that an enclosure at a uniform temperature above absolute zero always emits radiation as a result of atomic and molecular agitation (Ditchburn, 1963). Because of the uniform temperature, this radiation reaches an equilibrium value for the total energy density and has a definite distribution of energy with wavelength. It has been found that the total energy, $I(T)_{0-\infty}$, emitted over all wavelengths in a unit time from a unit area of the wall on the enclosure is a function of its temperature only and is

$$I(T)_{0-\infty} = \sigma T^4 ,$$

where T is the absolute temperature in degrees Kelvin. This is known as the Stefan-Boltzmann Fourth Power Law and σ the Stefan-Boltzmann constant (σ has been measured experimentally and theoretically; its derivation is discussed in a later section).

The distribution of energy is described by the function, $B_{\lambda,T}$, and we say that the energy in the wavelength range λ to $(\lambda + d\lambda)$ is $B_{\lambda,T}\, d\lambda$.

It has also been found experimentally that $B_{\lambda,T}$ has a maximum for a given value of absolute temperature T, at wavelength λ_m, and that this maximum is proportional to the fifth power of T, that is

$$B(T)_{max} = bT^5 , \tag{1}$$

where b is the proportionality constant. The wavelength where this maximum occurs has a definite relationship for all temperatures:

$$\lambda_m T = \text{constant} . \tag{2}$$

It is evident that these maximums shift toward the shorter wavelengths as the temperature rises.

Both of the above laws can be derived from Wien's displacement law, which says that the distribution of energy with wavelength obeys the following relationship:

$$B_{\lambda,T} = \lambda^{-5} F\left(\frac{1}{\lambda T}\right) .$$

Wien's displacement law, based on experimental thermodynamics (Ditchburn, 1963), can be used to determine Equations 1 and 2 (see Appendix A) but cannot be used to solve for the relationship between temperature, wavelength, and energy intensity. Wien later obtained an expression for $B_{\lambda,T}$, which agrees experimentally for short wavelengths beyond the ultraviolet,

$$B_{\lambda,T} = \frac{2\pi hc^2 \lambda^{-5}}{e^{hc/\lambda kT}} , \qquad \text{(Wien's Law)}$$

Rayleigh and Jeans, applying the laws of classical mechanics, obtained an expression that agrees well with experiments in the longer wavelength region

$$B_{\lambda,T} = \frac{2\pi c}{\lambda^4} kT , \qquad \text{(Rayleigh and Jeans' Law)}$$

but it was not until 1901 that Planck, using a new quantum hypothesis, was able to obtain an expression that was accurate over the whole length of the black-body spectrum. Figure 1 shows the back-body spectrum for T = 900°K, 1000°K, and 1100°K. Planck assumed that the possible energies of a mode of variation or frequency were all multiples of a basic vibration energy, $\epsilon_f = hf = hc/\lambda$, that is, 0, ϵ_f, $2\epsilon_f$, $3\epsilon_f$, $\cdots n\epsilon_f$. According to Boltzmann statistics, the probabilities of a given vibration with energies 0, ϵ_f, $2\epsilon_f$, $\cdots n\epsilon_f$ are in the ratios

$$1 ; \; e^{-\epsilon_f/kT} ; \; e^{-2\epsilon_f/kT} ; \; e^{-3\epsilon_f/kT} ; \; \cdots$$

and the most probable mean energy is

$$\bar{\epsilon}_f = \frac{\sum_{n=0}^{\infty} n\epsilon_f \, e^{-n\epsilon_f/kT}}{\sum_{n=0}^{\infty} e^{-n\epsilon_f/kT}}$$

$$= \frac{\epsilon_f}{e^{\epsilon_f/kT} - 1} \, .$$

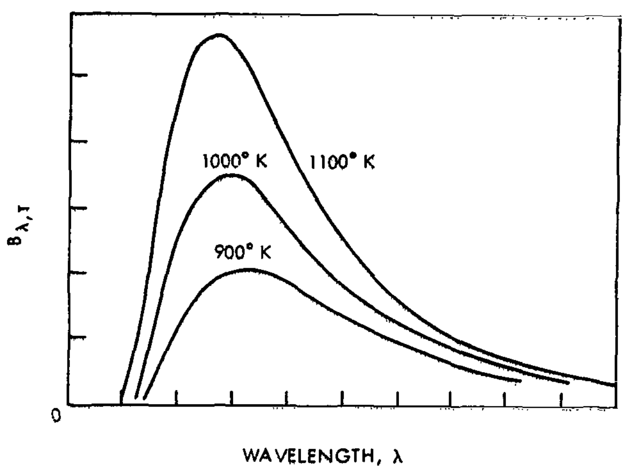

Figure 1—$B_{\lambda,T}$ vs. λ for T = 900°K, 1000°K, 1100°K.

Inserting this energy in Rayleigh and Jeans' Law in place of the energy kT gives

$$B_{\lambda,T} = \frac{2\pi c}{\lambda^4} \frac{\epsilon_f}{\left(e^{\epsilon_f/kT} - 1\right)}$$

or

$$B_{\lambda,T} = \frac{2\pi hc^2 \lambda^{-5}}{\left(e^{hc/\lambda kT} - 1\right)} \, , \qquad \text{(Planck's Law)}$$

which is Planck's Black-Body Radiation Law, more commonly referred to as the Planck Function (see Figure 1).

For convenience, the constants are grouped together to form the 1st and 2nd radiation constants c_1 and c_2,

$$c_1 = 2\pi hc^2 \qquad (\text{watts cm}^2)$$

$$= 2hc^2 \qquad (\text{watts cm}^2 \ \text{ster}^{-1})$$

and

$$c_2 = \frac{hc}{k} \qquad (\text{cm °K}) \, ,$$

such that

$$B_{\lambda,T} = \frac{c_1 \lambda^{-5}}{\left(e^{c_2/\lambda T} - 1\right)} \, .$$

These constants are given in Appendix B.

This paper refers to the Planck function in two forms: in terms of wavelength λ and in terms of wave number ν $(= \lambda^{-1})$.

PLANCK FUNCTION

As stated previously, the Planck function (which is accepted as the correct expression for the relationship between black-body intensity, wavelength, and temperature) is[1]

$$B_{\lambda,T} = \frac{c_1 \lambda^{-5}}{e^{c_2/\lambda T} - 1} ,$$

or, for the total intensity between λ and $(\lambda + d\lambda)$,

$$B_{\lambda,T}\, d\lambda = \frac{c_1 \lambda^{-5}}{e^{c_2/\lambda T} - 1}\, d\lambda .$$

Writing in terms of wave number ν, where

$$\lambda = \nu^{-1}$$

and

$$d\lambda = -\frac{d\nu}{\nu^2} ,$$

we have

$$B_{\nu,T} = \frac{c_1 \nu^5}{e^{c_2 \nu/T} - 1} \tag{3}$$

and

$$B_{\nu,T}\, d\nu = -B_{\lambda,T}\, d\lambda = \frac{c_1 \nu^3}{e^{c_2 \nu/T} - 1}\, d\nu . \tag{4}$$

NORMALIZED PLANCK FUNCTION

It is important to have a picture of relative black-body spectrums for various temperatures, not only to see relative magnitudes but also to see the shape of the spectrum over various ranges of wave number. A convenient way of showing this without an excess of graphs for any chosen temperature range is to find a normalizing function such that the shapes of the relative spectrums are not changed considerably. Normalizing by the maximum intensity accomplishes this.

Differentiating $B_{\nu,T}$ with respect to ν gives

$$\frac{dB_{\nu,T}}{d\nu} = \frac{c_1 \nu^4}{\left(e^{c_2\nu/T} - 1\right)} \left[5 - \frac{c_2 \nu}{T} \left(1 - e^{-c_2\nu/T}\right)^{-1}\right].$$

For a maximum, the terms in the brackets are set to zero and can be solved by successive approximation to obtain

$$\frac{c_2 \nu}{T} = 4.96511 \ , \tag{5}$$

which is correct to six figures (Harrison, 1960). For a given temperature, this gives the wave number ν_m at which maximum intensity occurs:

$$\nu_m = \frac{4.96511 \cdot T}{c_2} \ . \tag{6}$$

Substituting Equations 5 and 6 into Equation 3 gives the maximum intensity occurring at ν_m:

$$B(T)_{max} = c_1 \frac{\left(\frac{4.96511 \cdot T}{c_2}\right)^5}{e^{4.96511} - 1}$$

or

$$B(T)_{max} = bT^5 \text{ watts cm}^{-3} \text{ ster}^{-1} \ ,$$

where the proportionality constant b for maximum intensity is

$$b = 21.202 \frac{c_1}{c_2^5}$$

$$= 4.09512 \times 10^{-12} \text{ watts cm}^{-3} \text{ ster}^{-1} \text{ deg.}^{-5} \ .$$

This leads to a convenient normalized function using $B(T)_{max}$ for normalization, which will be referred to as $B(\nu/T)_{Norm}$, or

$$B\left(\frac{\nu}{T}\right)_{Norm} = \frac{B_{\nu,T}}{B(T)_{max}} = \frac{c_1}{b} \frac{\left(\frac{\nu}{T}\right)^5}{e^{c_2\nu/T} - 1} .$$

$B(\nu/T)_{Norm}$ has been tabulated as a function of ν/T in Table 1. To obtain $B_{\nu,T}$, first interpolate to obtain $B(\nu/T)_{Norm}$; then multiply by bT^5:

$$B_{\nu,T} = bT^5 \cdot B\left(\frac{\nu}{T}\right)_{Norm} = B(T)_{max} \cdot B\left(\frac{\nu}{T}\right)_{Norm} .$$

$B(T)_{max}$ has been tabulated versus T in °K in Appendix C.

The plots on the graph in Figure 2 show the relative spectrums for temperature T_1 versus a maximum temperature T_2. Obviously T_2 is completely arbitrary so that this graph can be used for any temperature range. The functions plotted are

$$\frac{B_{\nu,T_1}}{B(T_2)_{max}} = \left(\frac{T_1}{T_2}\right)^5 \frac{B_{\nu,T_1}}{B(T_1)_{max}} = \left(\frac{T_1}{T_2}\right)^5 B\left(\frac{\nu}{T_1}\right)_{Norm}$$

for $T_1/T_2 = 1$, 0.9, 0.8, 0.7, 0.6, and 0.5. Thus, a fairly comprehensive picture of the relative black-body spectrums of sources over a wide temperature range is made possible. The dashed line through the centers is a plot of ν_m for the maximums falling on the graph.

TOTAL INTENSITY

To compute the total black-body intensity for a given temperature over a finite range of wavelengths, say from λ_1 to λ_2, it is necessary to compute the integral

$$I(T)_{\lambda_1-\lambda_2} = \int_{\lambda_1}^{\lambda_2} B_{\lambda,T}\, d\lambda$$

$$= \int_0^{\lambda_2} B_{\lambda,T}\, d\lambda - \int_0^{\lambda_1} B_{\lambda,T}\, d\lambda ,$$

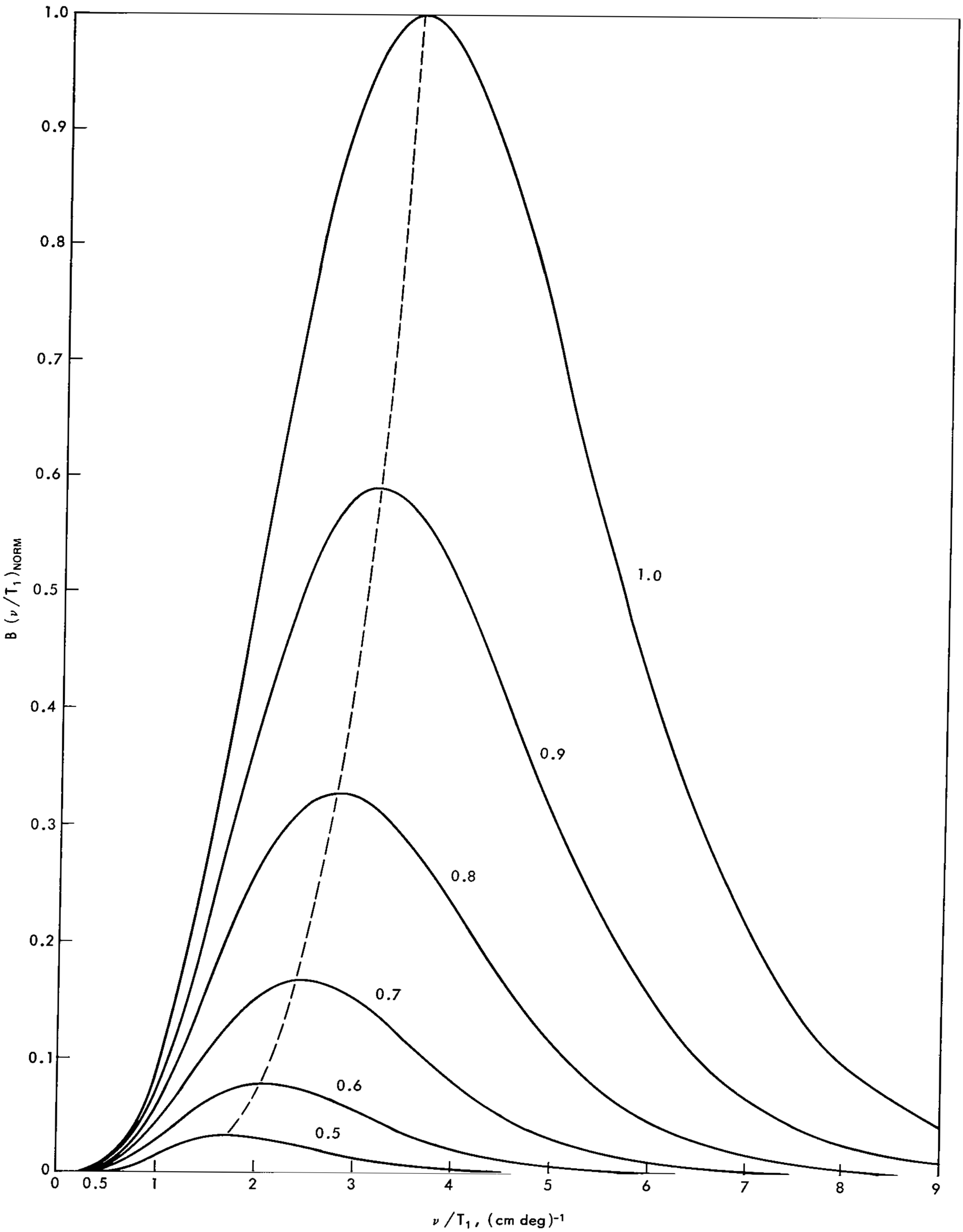

Figure 2—Normalized Planck function $(T_1/T_2)^5 B(\nu/T_1)_{Norm}$ for T_1/T_2 = 1.0, 0.9, 0.8, 0.7, 0.6, 0.5.

or in terms of wave number

$$I(T)_{\nu_1-\nu_2} = -\int_{\nu_1}^{\nu_2} B_{\nu,T}\, d\nu$$

$$= \int_{\nu_2}^{\nu_1} B_{\nu,T}\, d\nu$$

or

$$I(T)_{\nu_1-\nu_2} = \int_{\nu_2}^{\infty} B_{\nu,T}\, d\nu - \int_{\nu_1}^{\infty} B_{\nu,T}\, d\nu \ .$$

Therefore, working, say, in some field of pyrometry or spectroscopy, requires a convenient solution to the following integral:

$$I(T)_{\nu-\infty} = \int_{\nu}^{\infty} B_{\nu,T}\, d\nu = \int_{\nu}^{\infty} \frac{c_1 \nu^3}{e^{c_2\nu/T} - 1}\, d\nu \ . \tag{7}$$

Multiplying both numerator and denominator of the right-hand side of Equation 7 by $e^{-c_2\nu/T}$ gives

$$I(T)_{\nu-\infty} = \int_{\nu}^{\infty} \frac{c_1 \nu^3 e^{-c_2\nu/T}}{1 - e^{-c_2\nu/T}}\, d\nu \ . \tag{8}$$

Because

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$$

for

$$r^2 < 1 \ ,$$

and because ν and T are always greater than 0, Equation 8 can be written

$$I(T)_{\nu-\infty} = c_1 \int_{\nu}^{\infty} \nu^3 e^{-c_2\nu/T} \left(\sum_{n=0}^{\infty} e^{-nc_2\nu/T} \right) d\nu$$

$$= c_1 \int_{\nu}^{\infty} \nu^3 \left(\sum_{n=1}^{\infty} e^{-nc_2\nu/T} \right) d\nu \ . \tag{9}$$

As compared with the series e^{-n}, the series under the integral is uniformly and absolutely convergent; therefore, the integration can be performed term by term, and Equation 9 becomes

$$I(T)_{\nu-\infty} = c_1 \sum_{n=1}^{\infty} \int_{\nu}^{\infty} \nu^3 e^{-nc_2\nu/T} d\nu .$$

The integral can be integrated by parts to give

$$\int_{\nu}^{\infty} \nu^3 e^{-nk\nu} d\nu = e^{-nk\nu} \left[\frac{\nu^3}{nk} + 3 \frac{\nu^2}{(nk)^2} + 6 \frac{\nu}{(nk)^3} + \frac{6}{(nk)^4} \right] \tag{10}$$

or

$$\int_{\nu}^{\infty} \nu^3 e^{-nk\nu} d\nu = \frac{e^{-nk\nu}}{(nk)^4} \left[(nk\nu)^3 + 3(nk\nu)^2 + 6\, nk\nu + 6 \right] , \tag{11}$$

where, for convenience, $k = c_2/T$ (cm). Therefore,

$$I(T)_{\nu-\infty} = \frac{c_1}{k^4} \sum_{n=1}^{\infty} \frac{e^{-nk\nu}}{n^4} \left[(nk\nu)^3 + 3(nk\nu)^2 + 6\, nk\nu + 6 \right] . \tag{12}$$

For most values of T and ν, Equation 12 converges very rapidly; often to 3-digit accuracy in one or two terms (see Table 1).

Table 1

ν/T in (cm °K)$^{-1}$ Versus N, the Number of Terms Necessary in

$$\sum_{n=1}^{N} \frac{e^{-nc_2\nu/T}}{n^4} \left[\left(nc_2 \frac{\nu}{T}\right)^3 + 3\left(nc_2 \frac{\nu}{T}\right)^2 + 6\left(nc_2 \frac{\nu}{T}\right) + 6 \right]$$

for 2, 3, 4, and 5 Significant Digits of Accuracy in the Partial Sum.

ν/T Range (cm °K)$^{-1}$	N, the Number of Terms Necessary for Accuracies of			
	2 Sig. Digits	3 Sig. Digits	4 Sig. Digits	5 Sig. Digits
0.001—0.1	3	5	10	31
0.1—0.2	3	5	9	26
0.2—0.3	3	5	8	20
0.3—0.45	3	5	7	16
0.45—0.9	2	4	6	10
0.9—2.1	2	3	4	6
2.1—4.2	1	2	3	4
4.2—7.4	1	1	2	2
7.4—∞	1	1	1	1

Table 1 gives the number of terms, N, necessary in the summation to obtain at least the number of figures of accuracy stated for the ranges of (ν/T), such that

$$I(T)_{\nu-\infty} \cong \frac{c_1}{k^4} \sum_{n=1}^{N} \frac{e^{-nk\nu}}{n^4} \left[(nk\nu)^3 + 3(nk\nu)^2 + 6\, nk\nu + 6\right] . \tag{13}$$

In comparison, integrating Wien's Law

$$B_{\nu,T} = \frac{c_1 \nu^5}{e^{c_2\nu/T}} ,$$

gives

$$I(T)_{\nu-\infty} = \frac{c_1}{k^4} e^{-k\nu}\left[(k\nu)^3 + 3(k\nu)^2 + 6\, k\nu + 6\right] .$$

This is the same as Equation 12 with $n = 1$ and implies that Wien's Law is a limiting function for the Planck functions for values of ν/T where $N = 1$ in Table 1, or that Wien's Law actually depends on large values of ν/T rather than ν, as previously stated. Table 1 can then be used to define the dividing line for the application of Wien's Law.

STEFAN-BOLTZMANN CONSTANT

Immediately, using Equation 12, it is possible to compute the Stefan-Boltzmann constant, σ, the proportionality constant for the total intensity over all wavelengths or wave numbers. That is

$$I(T)_{0-\infty} = -\int_0^\infty B_{\nu,T}\, d\nu = \int_0^\infty B_\lambda\, d\lambda = \sigma T^4 .$$

Setting $\nu = 0$ in Equation 12 gives

$$I(T)_{0-\infty} = \int_0^\infty B_{\nu,T}\, d\nu$$

$$= \frac{6\, c_1}{k^4} \sum_{n=1}^{\infty} \frac{1}{n^4} = \sigma T^4 .$$

From Knopp, 1956 p. 173,

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} .$$

Replacing k by c_2/T gives

$$\sigma = \frac{c_1 \pi^4}{15 c_2^4} .$$

Substitution of c_1 and c_2 by their values in Appendix B gives the following value for the Stefan-Boltzmann constant:

$$\sigma = 1.80466 \times 10^{-12} \text{ watts cm}^{-2} \, {}^\circ\text{K}^{-4} \text{ ster}^{-1} .$$

NORMALIZED TOTAL INTENSITY

Again, as with the Planck function, it is desirable to have a convenient table or graph covering a wide range of temperatures and wave numbers but compact in form; like $B(T)_{max}$, the total intensity $I(T)_{0-\infty} = \sigma T^4$ performs this function. That is:

$$I\left(\frac{\nu}{T}\right)_{Norm} = \frac{I(T)_{\nu-\infty}}{I(T)_{0-\infty}} = \frac{\int_{\nu}^{\infty} B_{\nu,T} \, d\nu}{\sigma T^4} .$$

Equation 12 gives

$$I\left(\frac{\nu}{T}\right)_{Norm} = \frac{c_1}{\sigma c_2^4} \sum_{n=1}^{\infty} \frac{e^{-nk\nu}}{n^4} \left[(nk\nu)^3 + 3(nk\nu)^2 + 6\, nk\nu + 6\right] , \tag{14}$$

or, after replacing σ by $c_1 \pi^4/15 c_2^4$,

$$I\left(\frac{\nu}{T}\right)_{Norm} = \frac{15}{\pi^4} \sum_{n=1}^{\infty} \frac{e^{-nk\nu}}{n^4} \left[(nk\nu)^3 + 3(nk\nu)^2 + 6\, nk\nu + 6\right] ,$$

which, of course, is independent of the first radiation constant, c_1.

$I(\nu/T)_{Norm}$ has been tabulated versus ν/T in Appendix C; interpolating and multiplying by σT^4, gives

$$I(T)_{\nu-\infty} = \sigma T^4 \cdot I\left(\frac{\nu}{T}\right)_{Norm} = I(T)_{0-\infty} \cdot I\left(\frac{\nu}{T}\right)_{Norm} ,$$

$I(T)_{0-\infty}$ is tabulated versus T in °K in Appendix C.

Figure 3 shows the relative total intensities over a range of wave numbers for a very wide range of temperatures. The graphs represent the total intensity, from ν to ∞, emitted from a

source at temperature T_1, and can be used for comparison with a source at temperature T_2. That is, the functions plotted are

$$\frac{I(T_1)_{\nu-\infty}}{I(T_2)_{0-\infty}} = \left(\frac{T_1}{T_2}\right)^4 \frac{I(T_1)_{\nu-\infty}}{I(T_2)_{0-\infty}} = \left(\frac{T_1}{T_2}\right)^4 I\left(\frac{\nu}{T_1}\right)_{\text{Norm}}$$

for

$$\frac{T_1}{T_2} = 1.0,\ 0.9,\ 0.8,\ 0.7,\ 0.6,\ 0.5\ .$$

Clearly T_2 is arbitrary, so the graph can be used for any set of temperatures T_1/T_2 in the ranges given (see Figure 3).

REMAINDER ERROR ESTIMATE

Rewriting Equation 12 as

$$I(T)_{\nu-\infty} = \frac{c_1}{k^4} \sum_{n=1}^{N} \frac{e^{-nk\nu}}{n^4} \left[(nk\nu)^3 + 3(nk\nu)^2 + 6(nk\nu) + 6\right] + \epsilon_N$$

where ϵ_N is the remainder of the summation from $N + 1$ to ∞, or

$$\epsilon_N = \frac{c}{k^4} \sum_{N+1}^{\infty} \frac{e^{-nk\nu}}{n^4} \left[(nk\nu)^3 + 3(nk\nu)^2 + 6(nk\nu) + 6\right] ,$$

raises the question: what is an upper limit to the magnitude of ϵ_N?

Two methods are presented which give fairly good values for the ranges of ν and T investigated. In some cases one method is better than the other; therefore a criterion for usage is presented for those who want high accuracy in the integral, particularly in using the computer program presented in Appendix D.

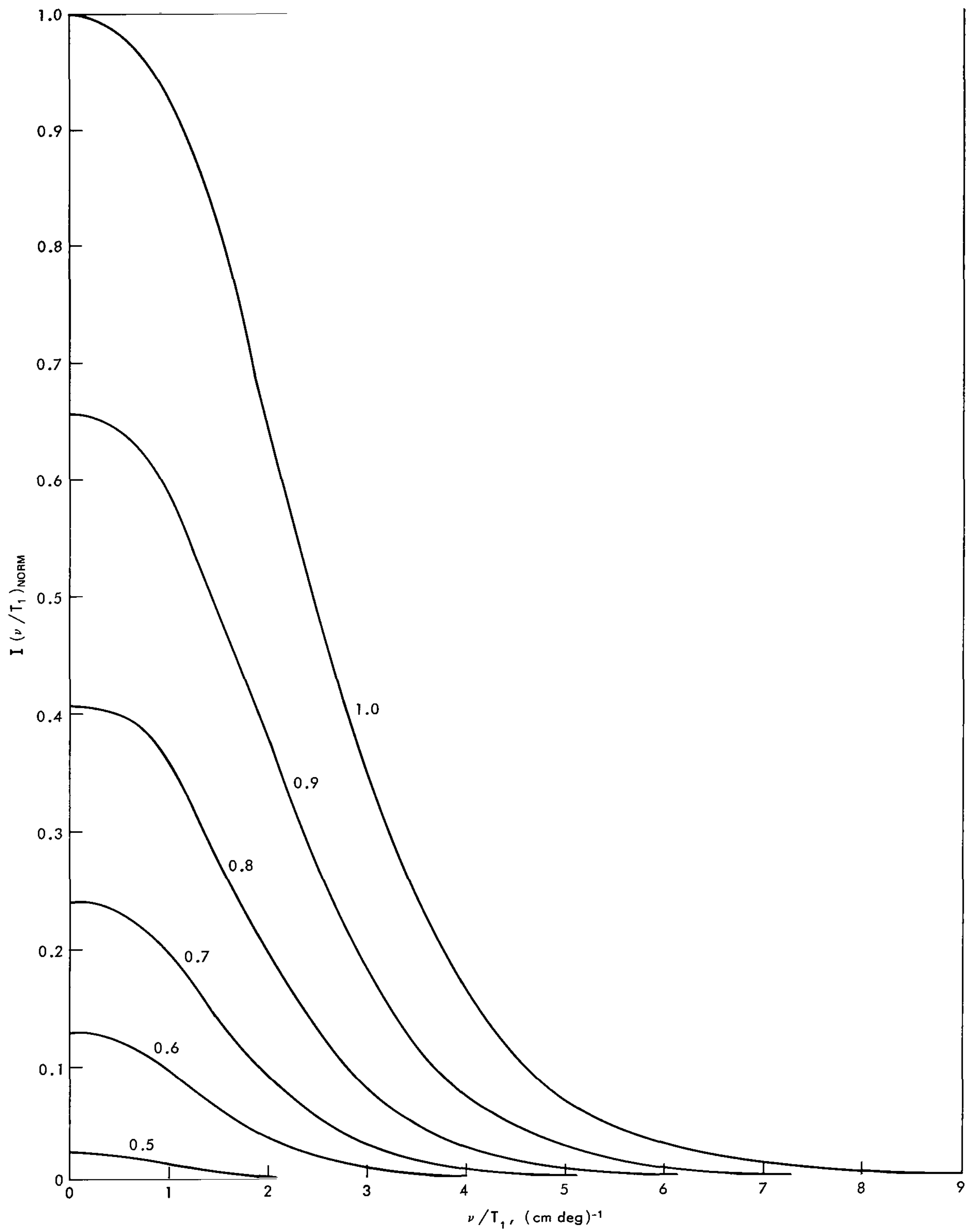

Figure 3—Normalized Planck integral $(T_1/T_2)^4 \, I(\nu/T_1)_{Norm}$ for T_1/T_2 = 1.0, 0.9, 0.8, 0.7, 0.6, 0.5.

In finding an upper bound, it was found convenient to rewrite Equation 15 as

$$\epsilon_N = c_1 \left(\frac{\nu^3}{k} \sum_{N+1}^{\infty} \frac{e^{-nk\nu}}{n} + \frac{3\nu^2}{k^2} \sum_{N+1}^{\infty} \frac{e^{-nk\nu}}{n^2} + \frac{6\nu}{k^3} \sum_{N+1}^{\infty} \frac{e^{-nk\nu}}{n^3} + \frac{6}{k^4} \sum_{N+1}^{\infty} \frac{e^{-nk\nu}}{n^4} \right)$$

and find an upper limit on the summations

$$U_{N_p} = \sum_{N+1}^{\infty} \frac{e^{-nk\nu}}{n^p} \qquad p = 1, 2, 3, 4 \quad . \tag{16}$$

SERIES METHOD

For convenience, let

$$U_{N_p} = \sum_{N+1}^{\infty} A_{n_p}(z) ;$$

where $z = k\nu$ and, as before, $k = c_2 / T$.

The ratio of the $(n + 1)$th term to the nth term is

$$\frac{A_{(n+1)_p}(z)}{A_{n_p}(z)} = \frac{e^{-(n+1)z}}{(n+1)^p} \frac{n^p}{e^{-nz}}$$

$$= \left(\frac{1}{1 + \frac{1}{n}} \right)^p e^{-z}$$

or

$$A_{(n+1)_p}(z) = A_{n_p}(z) \left(\frac{1}{1 + \frac{1}{n}} \right)^p e^{-z} \quad .$$

Since

$$A_n(z) \left(\frac{1}{1 + \frac{1}{n}} \right)^p \leq A_n(z) \quad ,$$

Therefore,

$$A_{(n+1)_p}(z) \leq A_{n_p}(z) e^{-z}$$

and

$$A_{(n+2)_p}(z) \leq A_{(n+1)_p}(z)\, e^{-z} = A_{n_p}\, e^{-2z} \quad ,$$

or, in general,

$$A_{(n+\ell)_p}(z) \leq A_{n_p}(z)\, e^{-\ell z} \; .$$

Therefore,

$$\sum_{\ell=0}^{\infty} A_{(n+\ell)_p}(z) \leq A_{n_p}(z) \sum_{\ell=0}^{\infty} e^{-\ell z}$$

or

$$\sum_{\ell=0}^{\infty} A_{(n+\ell)_p}(z) \leq \frac{A_{n_p}(z)}{1 - e^{-z}}$$

and

$$U_{N_p} = \sum_{N+1}^{\infty} \frac{e^{-nz}}{n^p} \leq \frac{A_{(N+1)_p}(z)}{1 - e^{-z}} \quad ,$$

giving the result

$$U_{N_p} \leq \frac{e^{-(N+1)z}}{(N+1)^p \left(1 - e^{-z}\right)} \quad . \tag{17}$$

This says that the error introduced by stopping the summation at some term N is less than or equal to the next term $(N + 1)$ divided by the factor $\left(1 - e^{-z}\right)$.

Upon substitution of Equation 17 into Equation 16, one obtains the expression for the upper limit on ϵ_N, that is,

$$\epsilon_N \leq \frac{c_1}{k^4} \frac{e^{-(N+1)z}}{(N+1)^4 \left(1 - e^{-z}\right)} \left\{\left[(N+1)\, k\nu\right]^3 + 3\left[(N+1)\, k\nu\right]^2 + 6\left[(N+1)\, k\nu\right] + 6\right\} \; . \tag{18}$$

SCHWARZ INEQUALITY METHOD

Another method, using the Schwarz inequality, seems worthwhile mentioning, since for very small values of $z\left(= c_1 \; \nu/T\right)$, Equation 18 would not yield a realistic limit, although it is certainly good enough for most cases.

The Schwarz Inequality says

$$\left(\sum a_n b_n\right)^2 \leq \sum a_n^2 \sum b_n^2 ,$$

provided a_n and b_n are at least piecewise continuous decreasing functions. (This, of course, holds in the present case, since $p \geq 1$.) Therefore,

$$\left(U'_{N_p}\right)^2 \leq \sum_{N+1}^{\infty} e^{-2nz} \cdot \sum_{N+1}^{\infty} \frac{1}{n^{2p}} .$$

Making the transformation

$$\ell = n - (N+1)$$

on the exponential term gives

$$\left(U'_{N_p}\right)^2 \leq e^{-2(N+1)z} \sum_{\ell=0}^{\infty} e^{-2\ell z} \cdot \sum_{N+1}^{\infty} \frac{1}{n^{2p}}$$

or

$$\left(U'_{N_p}\right)^2 \leq \frac{e^{-2(N+1)z}}{1 - e^{-2z}} \sum_{N+1}^{\infty} \frac{1}{n^{2p}} . \tag{19}$$

Table 2 contains values of

$$\sum_{n=1}^{N} \frac{1}{n^{2p}}$$

versus N for values of N from 1 to 20 and $p = 1, 2, 3, 4$. This can be used to determine the above summation from

$$\sum_{n=N+1}^{\infty} \frac{1}{n^{2p}} = \sum_{n=1}^{\infty} \frac{1}{n^{2p}} - \sum_{n=1}^{N} \frac{1}{n^{2p}} .$$

However, by letting $1/n^{2p} = g(n)$ and for

$$n \leq x \leq n + 1$$

Table 2

$\sum_{n=1}^{N} \frac{1}{n^{2p}}$ versus N for p = 1, 2, 3, 4.

N	$\sum_{n=1}^{N} \frac{1}{n^2}$	$\sum_{n=1}^{N} \frac{1}{n^4}$	$\sum_{n=1}^{N} \frac{1}{N^6}$	$\sum_{n=1}^{N} \frac{1}{N^8}$
1.0	1.0000000	1.0000000	1.0000000	1.0000000
2.0	1.2500000	1.0625000	1.0156250	1.0039063
3.0	1.3611111	1.0748457	1.0169967	1.0040587
4.0	1.4236111	1.0787519	1.0172409	1.0040739
5.0	1.4636111	1.0803519	1.0173049	1.0040765
6.0	1.4913889	1.0811235	1.0173263	1.0040771
7.0	1.5117970	1.0815400	1.0173348	1.0040772
8.0	1.5274220	1.0817842	1.0173386	1.0040773
9.0	1.5397677	1.0819366	1.0173405	1.0040773
10.0	1.5497677	1.0820366	1.0173415	1.0040773
11.0	1.55803216	1.0821049	1.0173421	1.0040773
12.0	1.5649766	1.0821531	1.0173424	1.0040773
13.0	1.57089375	1.0821881	1.0173426	1.0040773
14.0	1.57599579	1.0822141	1.0173427	1.0040773
15.0	1.5804402	1.0822339	1.0173428	1.0040773
16.0	1.5843465	1.0822491	1.0173428	1.0040773
17.0	1.5878067	1.0822611	1.0173429	1.0040773
18.0	1.5908931	1.0822706	1.0173429	1.0040773
19.0	1.5936632	1.08227828	1.0173429	1.0040773
20.0	1.5961632	1.0822845	1.0173429	1.0040773
∞	1.6449341	1.08232323	1.0173431	1.00407735

and

$$g(n) \geq g(x) \geq g(n+1) \ ,$$

integration may be performed over the unit interval from n to $n + 1$, and therefore

$$g(n) \geq \int_{n}^{n+1} g(x)\,dx \geq g(n+1)$$

or

$$\sum_{N+1}^{\infty} g(n) \geq \int_{N+1}^{\infty} g(x)\, dx \geq \sum_{N+1}^{\infty} g(n+1) = \sum_{N+1}^{\infty} g(n) - g(n+1) .$$

Hence

$$\int_{N+1}^{\infty} g(x)\, dx \leq \sum_{N+1}^{\infty} g(n) \leq \int_{N+1}^{\infty} g(x)\, dx + g(N+1)$$

Inserting

$$\int_{N+1}^{\infty} g(x)\, dx = (2p-1)\,(N+1)^{1-2p} .$$

gives

$$\sum_{N+1}^{\infty} \frac{1}{n^{2p}} \leq (N+1)^{-2p} \left(\frac{N+1}{2p-1} + 1 \right) . \tag{20}$$

Substitution of Equation 20 in Equation 19 gives

$$U_{N_p}' \leq \frac{e^{-(N+1)z}}{(N+1)^p} \left(\frac{N+1}{2p-1} + 1 \right)^{1/2} \left(1 - e^{-2z}\right)^{1/2} .$$

This says that the error introduced by stopping the summation in Equation 16 at the Nth term is less or equal to the (N + 1)th term reduced by the factor

$$\left(\frac{N+1}{2p-1} + 1 \right)^{1/2} \left(1 - e^{-2z}\right)^{-1/2} .$$

The error expression then takes the following form, using the Schwarz inequality:

$$\epsilon_N' \leq \frac{c_1}{k^4} \frac{e^{-(N+1)z}}{(N+1)^4 \left(1 - e^{-2z}\right)^{1/2}} \left\{ \left[(N+1)\, k\nu\right]^3 (N+2)^{1/2} + 3\left[(N+1)\, k\nu\right]^2 \left(\frac{N+4}{3}\right)^{1/2} + 6\left[(N+1)\, k\nu\right] \left(\frac{N+6}{5}\right)^{1/2} + 6\left(\frac{N+8}{7}\right)^{1/2} \right\} . \tag{21}$$

COMPARISON OF ϵ_N AND ϵ_N

Comparison of the expressions for ϵ_N and ϵ_N' is difficult; it is based on the question: which would give better bounds on the argument? The answer is: whichever one gives the smaller answer for a given N value. That is, the series expression, ϵ_N, if

$$\frac{\epsilon_N}{\epsilon_N'} < 1 \; . \tag{22}$$

or the Schwarz expression, ϵ_N', if

$$\frac{\epsilon_N'}{\epsilon_N} < 1 \; . \tag{23}$$

A quick evaluation of the two expressions shows that for $z > 1$ the series expression is the best, but, for $z \ll 1$ the Schwarz expression is best.

Using Equation 22, and taking the ratio for the $N - 1$ term gives:

$$\frac{\epsilon_{N-1}}{\epsilon_{N-1}'} = \left(\frac{1 + e^{-z}}{1 - e^{-z}}\right)^{1/2} \frac{\left[(Nk\nu)^3 + 3(Nk\nu)^2 + 6\,Nk\nu + 6\right]}{\left[(Nk\nu)^3\,(N+1)^{1/2} + 3(Nk\nu)^2\left(\frac{N+3}{3}\right)^{1/2} + 6\,Nk\nu\left(\frac{N+5}{5}\right)^{1/2} + 6\left(\frac{N+7}{7}\right)^{1/2}\right]} < 1 \; .$$

Thus the rule is: use ϵ_N (Equation 18) whenever the above inequality is true; use ϵ_N' (Equation 21) whenever it is false, i.e. when the expression exceeds one. When the expression equals one, use either. This is a laborious way to find the best error expression. The following criterion, provides a simpler computation.

Computing each summation for $p = 1, 2, 3,$ and 4 of the two methods, that is, say

$$\frac{U_{N_p}}{U_{N_p}'} < 1 \; ,$$

gives:

$$\frac{(2p - 1)}{(N + 2p)} \frac{\left(1 + e^{-z}\right)}{\left(1 - e^{-z}\right)} < 1$$

or

$$
e^{-z} < \begin{cases} \dfrac{N+1}{N+3}, & p = 1 \\[2ex] \dfrac{N+1}{N+7}, & p = 2 \\[2ex] \dfrac{N+1}{N+11}, & p = 3 \\[2ex] \dfrac{N+1}{N+15}, & p = 4 \end{cases} .
$$

That is, if e^{-z} is less than the four quantities on the right, then U_{N_p} is better for that particular summation than U'_{N_p}.

Hence, the following rule,

$$
\text{if} \qquad e^{-z} < \frac{N+1}{N+15} \qquad , \text{use } \epsilon_N \, ,
$$

$$
\text{if} \qquad e^{-z} > \frac{N+1}{N+15} \qquad , \text{use } \epsilon_N' \, ,
$$

$$
\text{if} \qquad e^{-z} = \frac{N+1}{N+15} \qquad , \text{use either } \epsilon_N \text{ or } \epsilon_N' \, .
$$

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Appendix A

Derivation of Equation 1 and 2

Given Wien's displacement law,

$$B_{\lambda,T} = \lambda^{-5} F\left(\frac{1}{\lambda T}\right) .$$

(a) To prove that total intensity proportional to T^4, first integrate

$$I(T)_{0-\infty} = \int_0^\infty B_{\lambda,T}\, d\lambda = \int_0^\infty \lambda^{-5} F\left(\frac{1}{\lambda T}\right) d\lambda .$$

Setting

$$\delta = T\lambda ,$$

$$d\delta = T\, d\lambda ,$$

gives

$$I(T)_{0-\infty} = T^4 \int_0^\infty \delta^{-5} F\left(\frac{1}{\delta}\right) d\delta .$$

Thus, total intensity is proportional to T^4.

(b) To obtain λ_m, first differentiate Wien's displacement law

$$\frac{dB_{\lambda,T}}{d\lambda} = -5\lambda_m^{-6} F\left(\frac{1}{\lambda T}\right) - \lambda_m^{-7} T^{-1} F'\left(\frac{1}{\lambda_m T}\right) .$$

and set equal to zero, i.e.,

$$(\lambda_m T)^{-1} F'\left(\frac{1}{\lambda_m T}\right) + 5F\left(\frac{1}{\lambda_m T}\right) = 0 .$$

If $x = 1/\lambda_m T$, and the differential equation

$$xf'(x) + 5f(x) = 0$$

has a solution, the solution must be x = constant. Therefore,

$$\lambda_m T = \text{constant}.$$

Appendix B

Black-Body Radiation Constants and Formulas

All values are consistent with those given in "Handbook for Chemistry and Physics."
(All constants refer to hemispherical radiation;
temperatures are always in degrees Kelvin.)

Constants

σ	Stefan-Boltzmann constant
	$= \dfrac{2\pi^5 k^4}{15 h^3 c^2} = \dfrac{c_1 \pi^4}{15 c_2^4}$
	$= 1.80466 \times 10^{-12}$ watts cm^{-2} deg^{-4} $ster^{-1}$
c	velocity of light
	$= 2.997925 \times 10^{10}$ cm sec^{-1}
c_1	1st radiation constant
	$= 2c^2 h$
	$= 1.1909 \times 10^{-12}$ watts cm^2 ster
c_2	2nd radiation constant
	$= \dfrac{hc}{k}$
	$= 1.43879$ cm deg
h	Planck's constant
	$= 6.6252 \times 10^{-34}$ watt sec^2
k	Boltzmann constant
	$= 1.38044 \times 10^{-23}$ watt sec deg^{-1}
b	Maximum intensity constant
	$= 21.202 \dfrac{c_1}{c_2^5}$
	$= 4.09512 \times 10^{-12}$ watts cm^{-3} deg^{-5} $ster^{-1}$

Formulas (in all formulas ν = wave number = λ^{-1})

1. Planck function or black-body intensity

$$B_{\nu,T} = \frac{c_1 \nu^5}{e^{c_2\nu/T} - 1} \text{ watts cm}^{-3} \text{ ster}^{-1}$$

2. Maximum intensity

$$B(T)_{max} = bT^5 \text{ watts cm}^{-3} \text{ ster}^{-1}$$

3. Normalized intensity

$$B\left(\frac{\nu}{T}\right)_{Norm} = \frac{B_{\nu,T}}{B(T)_{max}} = \frac{c_1}{b} \frac{\left(\frac{\nu}{T}\right)^5}{e^{c_2\nu/T} - 1} = \frac{21.202}{c_2^5} \frac{\left(\frac{\nu}{T}\right)^5}{\left(e^{c_2\nu/T} - 1\right)}$$

$$\frac{B_{\nu,T_1}}{B(T_2)_{max}} = \left(\frac{T_1}{T_2}\right)^5 B\left(\frac{\nu}{T_1}\right)_{Norm}$$

4. Total intensity $\nu = 0 - \infty$

$$I(T)_{0-\infty} = \sigma T^4 \text{ watts cm}^{-2} \text{ ster}^{-1}$$

5. Total intensity from $\nu - \infty$

$$I(T)_{\nu-\infty} = \int_\nu^\infty B_{\nu,T}\, d\nu = \int_0^\lambda B_{\lambda,T}\, d\lambda$$

$$= \frac{c_1}{k'^4} \sum_{n=1}^{\infty} \frac{e^{-nk'\nu}}{n^4} \left[(nk'\nu)^3 + 3(nk'\nu)^2 + 6\, nk'\nu + 6\right]$$

$$k' = \frac{c_2}{T}$$

6. Normalized total intensity

$$I\left(\frac{\nu}{T}\right)_{Norm} = \frac{I(T)_{\nu-\infty}}{I(T)_{0-\infty}} = \frac{I(T)_{\nu-\infty}}{\sigma T^4}$$

$$\frac{I(T_1)_{\nu-\infty}}{I(T_2)_{0-\infty}} = \left(\frac{T_1}{T_2}\right)^4 I\left(\frac{\nu}{T_1}\right)_{Norm}$$

$$I\left(\frac{\nu}{T}\right)_{Norm} = \frac{15}{\pi^4} \sum_{n=1}^{\infty} \frac{e^{-nk'\nu}}{n^4} \left[(nk'\nu)^3 + 3(nk'\nu)^2 + 6(nk'\nu) + 6\right]$$

$$k' = \frac{c_2}{T}$$

7. Wave number where $B(T)_{max}$ occurs

$$\nu_m = \frac{4.96511}{c_2} T$$

$$= 3.4509 \, T$$

Appendix C

Tables of Normalized Planck Functions, Planck Integrals, Total Intensity, and Maximum Intensity

(a) Table C1 contains tables of normalized Planck functions, $B(\nu/T)_{Norm}$, where

$$B\left(\frac{\nu}{T}\right)_{Norm} = \frac{B_{\nu,T}}{bT^5} = 21.202\,\frac{\left(\frac{\nu}{T}\right)^5}{\left(e^{c_2\nu/T} - 1\right)}$$

and normalized Planck integrals, $I(\nu/T)_{Norm}$, where

$$I\left(\frac{\nu}{T}\right)_{Norm} = \frac{\int_{\nu}^{\infty} B_{\nu,T}\,d\nu}{\sigma T^4} = 15\left(\frac{c_2}{\pi T}\right)^4 \int_{\nu}^{\infty} \frac{\nu^3}{\left(e^{c_2\nu/T} - 1\right)}\,d\nu$$

versus ν/T in (cm °K)$^{-1}$

T = temperature in °K

ν = wave number ($= \lambda^{-1}$) cm^{-1}

c_2 = 2nd radiation constant = 1.43879 cm °K

c_1 = does not enter into these tables

σ = Stefan-Boltzmann constant = 1.80466×10^{-12} watts cm^{-2} deg^{-4} ster^{-1}.

b) Table C2 contains tables of total intensity, $I(T)_{0-\infty}$, and maximum intensity, $B(T)_{max}$, where

$$I(T)_{0-\infty} = \int_0^{\infty} B_{\nu,T}\,d\nu = \sigma T^4$$

$$B(T)_{max} = bT^5$$

versus T in °K. $b = 4.09512 \times 10^{-12}$ watts cm^{-3} deg^{-5} ster^{-1}.

Table C1

ν/T in $(\mathrm{cm}\ {}^{\circ}\mathrm{K})^{-1}$ vs $B(\nu/T)_{Norm}$ and $I(\nu/T)_{Norm}$.

$c_1 = 1.43879$ cm °K; c_2 does not enter.

$\frac{\nu}{T}$	$B\left(\frac{\nu}{T}\right)_{Norm}$	$I\left(\frac{\nu}{T}\right)_{Norm}$	$\frac{\nu}{T}$	$B\left(\frac{\nu}{T}\right)_{Norm}$	$I\left(\frac{\nu}{T}\right)_{Norm}$
0.001	2.0194E-13	1.0000E 00	2.900	9.3381E-01	3.7201E-01
0.002	3.2290E-12	1.0000E 00	3.000	9.5604E-01	3.4735E-01
0.003	1.6336E-11	1.0000E 00	3.100	9.7366E-01	3.2380E-01
0.004	5.1595E-11	1.0000E 00	3.200	9.8669E-01	3.0138E-01
0.005	1.2587E-10	1.0000E 00	3.300	9.9524E-01	2.8007E-01
0.006	2.6083E-10	1.0000E 00	3.400	9.9944E-01	2.5990E-01
0.007	4.8286E-10	1.0000E 00	3.500	9.9949E-01	2.4083E-01
0.008	8.2314E-10	1.0000E 00	3.600	9.9560E-01	2.2286E-01
0.009	1.3176E-09	1.0000E 00	3.700	9.8802E-01	2.0596E-01
0.010	2.0067E-09	1.0000E 00	3.800	9.7702E-01	1.9010E-01
0.020	3.1876E-08	1.0000E 00	3.900	9.6289E-01	1.7524E-01
0.030	1.6021E-07	1.0000E 00	4.000	9.4592E-01	1.6135E-01
0.040	5.0268E-07	9.9999E-01	4.100	9.2641E-01	1.4839E-01
0.050	1.2184E-06	9.9998E-01	4.200	9.0466E-01	1.3633E-01
0.060	2.5080E-06	9.9997E-01	4.300	8.8096E-01	1.2511E-01
0.070	4.6126E-06	9.9995E-01	4.400	8.5560E-01	1.1469E-01
0.080	7.8115E-06	9.9993E-01	4.500	8.2886E-01	1.0503E-01
0.090	1.2421E-05	9.9989E-01	4.600	8.0100E-01	9.6099E-02
0.100	1.8793E-05	9.9986E-01	4.700	7.7227E-01	8.7841E-02
0.200	2.7909E-04	9.9891E-01	4.800	7.4290E-01	8.0219E-02
0.300	1.3092E-03	9.9650E-01	4.900	7.1312E-01	7.3194E-02
0.400	3.8274E-03	9.9217E-01	5.000	6.8312E-01	6.6727E-02
0.500	8.6288E-03	9.8555E-01	5.100	6.5308E-01	6.0781E-02
0.600	1.6495E-02	9.7645E-01	5.200	6.2317E-01	5.5320E-02
0.700	2.8126E-02	9.6474E-01	5.300	5.9354E-01	5.0311E-02
0.800	4.4087E-02	9.5041E-01	5.400	5.6432E-01	4.5720E-02
0.900	6.4784E-02	9.3351E-01	5.500	5.3563E-01	4.1518E-02
1.000	9.0438E-02	9.1416E-01	5.600	5.0755E-01	3.7675E-02
1.100	1.2109E-01	8.9252E-01	5.700	4.8019E-01	3.4164E-02
1.200	1.5659E-01	8.6881E-01	5.800	4.5360E-01	3.0959E-02
1.300	1.9664E-01	8.4325E-01	5.900	4.2785E-01	2.8037E-02
1.400	2.4079E-01	8.1609E-01	6.000	4.0299E-10	2.5374E-02
1.500	2.8847E-01	7.8758E-01	6.100	3.7904E-01	2.2949E-02
1.600	3.3901E-01	7.5799E-01	6.200	3.5604E-01	2.0744E-02
1.700	3.9170E-01	7.2757E-01	6.300	3.3401E-01	1.8740E-02
1.800	4.4576E-01	6.9656E-01	6.400	3.1294E-01	1.6920E-02
1.900	5.0041E-01	6.6520E-01	6.500	2.9284E-01	1.5268E-02
2.000	5.5488E-01	6.3372E-01	6.600	2.7371E-01	1.3769E-02
2.100	6.0841E-01	6.0231E-01	6.700	2.5554E-01	1.2412E-02
2.200	6.6033E-01	5.7117E-01	6.800	2.3831E-01	1.1182E-02
2.300	7.0997E-01	5.4045E-01	6.900	2.2200E-01	1.0069E-02
2.400	7.5678E-01	5.1030E-01	7.000	2.0659E-01	9.0623E-03
2.500	8.0026E-01	4.8086E-01	7.100	1.9205E-01	8.1524E-03
2.600	8.3999E-01	4.5223E-01	7.200	1.7836E-01	7.3305E-03
2.700	8.7565E-01	4.2449E-01	7.300	1.6549E-01	6.5884E-03
2.800	9.0698E-01	3.9773E-01	7.400	1.5340E-01	5.9187E-03

Table C1 (Continued)

$\frac{\nu}{T}$	$B\left(\frac{\nu}{T}\right)_{Norm}$	$I\left(\frac{\nu}{T}\right)_{Norm}$	$\frac{\nu}{T}$	$B\left(\frac{\nu}{T}\right)_{Norm}$	$I\left(\frac{\nu}{T}\right)_{Norm}$
7.500	1.4207E-01	5.3149E-03	12.500	1.3722E-03	1.6433E-05
7.600	1.3145E-01	4.7706E-03	12.600	1.2367E-03	1.4555E-05
7.700	1.2152E-01	4.2802E-03	12.700	1.1141E-03	1.2889E-05
7.800	1.1225E-01	3.8387E-03	12.800	1.0034E-03	1.1413E-05
7.900	1.0360E-01	3.4414E-03	12.900	9.0342E-04	1.0103E-05
8.000	9.5543E-02	3.0840E-03	13.000	8.1315E-04	8.9428E-06
8.100	8.8041E-02	2.7627E-03	13.100	7.3169E-04	7.9144E-06
8.200	8.1067E-02	2.4739E-03	13.200	6.5820E-04	7.0032E-06
8.300	7.4590E-02	2.2145E-03	13.300	5.9191E-04	6.1958E-06
8.400	6.8580E-02	1.9816E-03	13.400	5.3215E-04	5.4808E-06
8.500	6.3010E-02	1.7725E-03	13.500	4.7830E-04	4.8476E-06
8.600	5.7853E-02	1.5850E-03	13.600	4.2977E-04	4.2869E-06
8.700	5.3081E-02	1.4169E-03	13.700	3.8606E-04	3.7904E-06
8.800	4.8671E-02	1.2661E-03	13.800	3.4671E-04	3.3510E-06
8.900	4.4599E-02	1.1311E-03	13.900	3.1128E-04	2.9622E-06
9.000	4.0841E-02	1.0101E-03	14.000	2.7940E-04	2.6180E-06
9.100	3.7377E-02	9.0179E-04	14.100	2.5073E-04	2.3136E-06
9.200	3.4186E-02	8.0485E-04	14.200	2.2494E-04	2.0443E-06
9.300	3.1249E-02	7.1812E-04	14.300	2.0175E-04	1.8061E-06
9.400	2.8548E-02	6.4055E-04	14.400	1.8091E-04	1.5954E-06
9.500	2.6066E-02	5.7119E-04	14.500	1.6218E-04	1.4091E-06
9.600	2.3786E-02	5.0920E-04	14.600	1.4536E-04	1.2444E-06
9.700	2.1694E-02	4.5381E-04	14.700	1.3025E-04	1.0989E-06
9.800	1.9776E-02	4.0434E-04	14.800	1.1669E-04	9.7019E-07
9.900	1.8017E-02	3.6017E-04	14.900	1.0451E-04	8.5647E-07
10.000	1.6407E-02	3.2074E-04	15.000	9.3582E-05	7.5600E-07
10.100	1.4933E-02	2.8555E-04	15.100	8.3779E-05	6.6723E-07
10.200	1.3585E-02	2.5416E-04	15.200	7.4987E-05	5.8882E-07
10.300	1.2353E-02	2.2616E-04	15.300	6.7102E-05	5.1956E-07
10.400	1.1227E-02	2.0120E-04	15.400	6.0034E-05	4.5839E-07
10.500	1.0199E-02	1.7896E-04	15.500	5.3699E-05	4.0438E-07
10.600	9.2606E-03	1.5913E-04	15.600	4.8023E-05	3.5669E-07
10.700	8.4051E-03	1.4147E-04	15.700	4.2937E-05	3.1459E-07
10.800	7.6253E-03	1.2574E-04	15.800	3.8383E-05	2.7743E-07
10.900	6.9149E-03	1.1173E-04	15.900	3.4305E-05	2.4463E-07
11.000	6.2680E-03	9.9267E-05	16.000	3.0653E-05	2.1569E-07
11.100	5.6793E-03	8.8172E-05	16.100	2.7386E-05	1.9015E-07
11.200	5.1439E-03	7.8301E-05	16.200	2.4462E-05	1.6762E-07
11.300	4.6570E-03	6.9520E-05	16.300	2.1846E-05	1.4774E-07
11.400	4.2146E-03	6.1712E-05	16.400	1.9506E-05	1.3021E-07
11.500	3.8127E-03	5.4769E-05	16.500	1.7413E-05	1.1474E-07
11.600	3.4479E-03	4.8598E-05	16.600	1.5542E-05	1.0110E-07
11.700	3.1167E-03	4.3114E-05	16.700	1.3870E-05	8.9076E-08
11.800	2.8164E-03	3.8241E-05	16.800	1.2375E-05	7.8474E-08
11.900	2.5441E-03	3.3913E-05	16.900	1.1039E-05	6.9125E-08
12.000	2.2973E-03	3.0069E-05	17.000	9.8461E-06	6.0885E-08
12.100	2.0737E-03	2.6656E-05	17.100	8.7806E-06	5.3623E-08
12.200	1.8713E-03	2.3626E-05	17.200	7.8288E-06	4.7221E-08
12.300	1.6880E-03	2.0937E-05	17.300	6.9791E-06	4.1580E-08
12.400	1.5222E-03	1.8550E-05	17.400	6.2206E-06	3.6610E-08

Table C1 (Continued)

$\frac{\nu}{T}$	$B\left(\frac{\nu}{T}\right)_{Norm}$	$I\left(\frac{\nu}{T}\right)_{Norm}$	$\frac{\nu}{T}$	$B\left(\frac{\nu}{T}\right)_{Norm}$	$I\left(\frac{\nu}{T}\right)_{Norm}$
17.500	5.5435E-06	3.2231E-08	22.500	1.4629E-08	5.0067E-11
17.600	4.9394E-06	2.8373E-08	22.600	1.2953E-08	4.3920E-11
17.700	4.4004E-06	2.4975E-08	22.700	1.1467E-08	3.8525E-11
17.800	3.9196E-06	2.1982E-08	22.800	1.0152E-08	3.3792E-11
17.900	3.4907E-06	1.9345E-08	22.900	8.9855E-09	2.9638E-11
18.000	3.1083E-06	1.7024E-08	23.000	7.9527E-09	2.5993E-11
18.100	2.7674E-06	1.4980E-08	23.100	7.0381E-09	2.2796E-11
18.200	2.4635E-06	1.3180E-08	23.200	6.2280E-09	1.9990E-11
18.300	2.1926E-06	1.1596E-08	23.300	5.5106E-09	1.7530E-11
18.400	1.9512E-06	1.0201E-08	23.400	4.8754E-09	1.5371E-11
18.500	1.7362E-06	8.9726E-09	23.500	4.3131E-09	1.3477E-11
18.600	1.5446E-06	7.8921E-09	23.600	3.8152E-09	1.1816E-11
18.700	1.3740E-06	6.9410E-09	23.700	3.3746E-09	1.0359E-11
18.800	1.2220E-06	6.1041E-09	23.800	2.9845E-09	9.0817E-12
18.900	1.0867E-06	5.3676E-09	23.900	2.6393E-09	7.9612E-12
19.000	9.6621E-07	4.7196E-09	24.000	2.3338E-09	6.9785E-12
19.100	8.5900E-07	4.1496E-09	24.100	2.0636E-09	6.1171E-12
19.200	7.6356E-07	3.6481E-09	24.200	1.8244E-09	5.3615E-12
19.300	6.7864E-07	3.2070E-09	24.300	1.6129E-09	4.6992E-12
19.400	6.0308E-07	2.8190E-09	24.400	1.4257E-09	4.1184E-12
19.500	5.3586E-07	2.4778E-09	24.500	1.2601E-09	3.6092E-12
19.600	4.7607E-07	2.1777E-09	24.600	1.1137E-09	3.1629E-12
19.700	4.2290E-07	1.9138E-09	24.700	9.8424E-10	2.7716E-12
19.800	3.7562E-07	1.6818E-09	24.800	8.6976E-10	2.4286E-12
19.900	3.3358E-07	1.4778E-09	24.900	7.6849E-10	2.1280E-12
20.000	2.9621E-07	1.2984E-09	25.000	6.7898E-10	1.8644E-12
20.100	2.6299E-07	1.1408E-09	25.100	5.9986E-10	1.6335E-12
20.200	2.3347E-07	1.0022E-09	25.200	5.2990E-10	1.4311E-12
20.300	2.0724E-07	8.8039E-10	25.300	4.6806E-10	1.2537E-12
20.400	1.8393E-07	7.7333E-10	25.400	4.1342E-10	1.0983E-12
20.500	1.6322E-07	6.7924E-10	25.500	3.6512E-10	9.6204E-13
20.600	1.4484E-07	5.9657E-10	25.600	3.2244E-10	8.4268E-13
20.700	1.2850E-07	5.2392E-10	25.700	2.8473E-10	7.3811E-13
20.800	1.1399E-07	4.6009E-10	25.800	2.5140E-10	6.4648E-13
20.900	1.0111E-07	4.0401E-10	25.900	2.2197E-10	5.6619E-13
21.000	8.9678E-08	3.5474E-10	26.000	1.9596E-10	4.9586E-13
21.100	7.9530E-08	3.1146E-10	26.100	1.7299E-10	4.3425E-13
21.200	7.0518E-08	2.7344E-10	26.200	1.5270E-10	3.8028E-13
21.300	6.2523E-08	2.4005E-10	26.300	1.3478E-10	3.3300E-13
21.400	5.5427E-08	2.1073E-10	26.400	1.1895E-10	2.9159E-13
21.500	4.9131E-08	1.8497E-10	26.500	1.0498E-10	2.5531E-13
21.600	4.3546E-08	1.6236E-10	26.600	9.2639E-11	2.2355E-13
21.700	3.8591E-08	1.4250E-10	26.700	8.1744E-11	1.9572E-13
21.800	3.4197E-08	1.2506E-10	26.800	7.2126E-11	1.7136E-13
21.900	3.0300E-08	1.0975E-10	26.900	6.3634E-11	1.5002E-13
22.000	2.6844E-08	9.6303E-11	27.000	5.6138E-11	1.3133E-13
22.100	2.3780E-08	8.4504E-11	27.100	4.9523E-11	1.1496E-13
22.200	2.1063E-08	7.4145E-11	27.200	4.3684E-11	1.0063E-13
22.300	1.8656E-08	6.5053E-11	27.300	3.8530E-11	8.8088E-14
22.400	1.6521E-08	5.7072E-11	27.400	3.3982E-11	7.7104E-14

Table C1 (Continued)

$\frac{\nu}{T}$	$B\left(\frac{\nu}{T}\right)_{Norm}$	$I\left(\frac{\nu}{T}\right)_{Norm}$	$\frac{\nu}{T}$	$B\left(\frac{\nu}{T}\right)_{Norm}$	$I\left(\frac{\nu}{T}\right)_{Norm}$
27.500	2.9969E-11	6.7485E-14	28.800	5.8165E-12	1.1900E-14
27.600	2.6428E-11	5.9065E-14	28.900	5.1251E-12	1.0411E-14
27.700	2.3304E-11	5.1694E-14	29.000	4.5156E-12	9.1069E-15
27.800	2.0549E-11	4.5240E-14	29.100	3.9784E-12	7.9665E-15
27.900	1.8117E-11	3.9591E-14	29.200	3.5048E-12	6.9685E-15
28.000	1.5972E-11	3.4646E-14	29.300	3.0875E-12	6.0954E-15
28.100	1.4081E-11	3.0318E-14	29.400	2.7197E-12	5.3315E-15
28.200	1.2412E-11	2.6529E-14	29.500	2.3955E-12	4.6631E-15
28.300	1.0941E-11	2.3213E-14	29.600	2.1099E-12	4.0784E-15
28.400	9.6434E-12	2.0311E-14	29.700	1.8583E-12	3.5670E-15
28.500	8.4991E-12	1.7771E-14	29.800	1.6365E-12	3.1195E-15
28.600	7.4904E-12	1.5548E-14	29.900	1.4412E-12	2.7281E-15
28.700	6.6007E-12	1.3602E-14	30.000	1.2690E-12	2.3857E-15

Table C2

$T(^\circ K)$ vs $I(T)_{0-\infty}$ and $B(T)_{max}$

$I(T)_{0-\infty} = \sigma T^4$ watts cm^{-2} ster.$^{-1}$

$B(T)_{max} = bT^5$ watts cm^{-3} ster.$^{-1}$

$T(^\circ K)$	$I(T)_{0-\infty}$	$B(T)_{max}$	$T(^\circ K)$	$I(T)_{0-\infty}$	$B(T)_{max}$
1.0	1.8047E-12	4.0951E-12	240.0	5.9874E-03	3.2608E 00
10.0	1.8047E-08	4.0951E-07	250.0	7.0494E-03	3.9991E 00
20.0	2.8875E-07	1.3104E-05	260.0	8.2469E-03	4.8655E 00
30.0	1.4618E-06	9.9510E-05	270.0	9.5907E-03	5.8760E 00
40.0	4.6199E-06	4.1934E-04	280.0	1.1092E-02	7.0478E 00
50.0	1.1279E-05	1.2797E-03	290.0	1.2764E-02	8.3995E 00
60.0	2.3388E-05	3.1843E-03	300.0	1.4618E-02	9.9510E 00
70.0	4.3330E-05	6.8826E-03	310.0	1.6666E-02	1.1724E 01
80.0	7.3919E-05	1.3419E-02	320.0	1.8923E-02	1.3741E 01
90.0	1.1840E-04	2.4181E-02	330.0	2.1402E-02	1.6026E 01
100.0	1.8047E-04	4.0951E-02	340.0	2.4116E-02	1.8606E 01
110.0	2.6422E-04	6.5952E-02	350.0	2.7081E-02	2.1508E 01
120.0	3.7421E-04	1.0190E-01	360.0	3.0311E-02	2.4761E 01
130.0	5.1543E-04	1.5205E-01	370.0	3.3822E-02	2.8397E 01
140.0	6.9328E-04	2.2024E-01	380.0	3.7630E-02	3.2447E 01
150.0	9.1361E-04	3.1097E-01	390.0	4.1750E-02	3.6948E 01
160.0	1.1827E-03	4.2940E-01	400.0	4.6199E-02	4.1934E 01
170.0	1.5073E-03	5.8144E-01	410.0	5.0995E-02	4.7444E 01
180.0	1.8945E-03	7.7379E-01	420.0	5.6156E-02	5.3519E 01
190.0	2.3519E-03	1.0140E 00	430.0	6.1698E-02	6.0201E 01
200.0	2.8875E-03	1.3104E 00	440.0	6.7640E-02	6.7534E 01
210.0	3.5097E-03	1.6725E 00	450.0	7.4002E-02	7.5566E 01
220.0	4.2275E-03	2.1105E 00	460.0	8.0803E-02	8.4343E 01
230.0	5.0502E-03	2.6357E 00	470.0	8.8062E-02	9.3919E 01

Table C2 (Continued)

T(°K)	$I(T)_{0-\infty}$	$B(T)_{max}$	T(°K)	$I(T)_{0-\infty}$	$B(T)_{max}$
480.0	9.5799E-02	1.0434E 02	1460.0	8.1999E 00	2.7166E 04
490.0	1.0404E-01	1.1568E 02	1480.0	8.6585E 00	2.9078E 04
500.0	1.1279E-01	1.2797E 02	1500.0	9.1361E 00	3.1097E 04
520.0	1.3195E-01	1.5570E 02	1520.0	9.6332E 00	3.3226E 04
540.0	1.5345E-01	1.8803E 02	1540.0	1.0150E 01	3.5470E 04
560.0	1.7748E-01	2.2553E 02	1560.0	1.0688E 01	3.7834E 04
580.0	2.0422E-01	2.6878E 02	1580.0	1.1247E 01	4.0323E 04
600.0	2.3388E-01	3.1843E 02	1600.0	1.1827E 01	4.2940E 04
620.0	2.6666E-01	3.7516E 02	1620.0	1.2430E 01	4.5692E 04
640.0	3.0277E-01	4.3971E 02	1640.0	1.3055E 01	4.8583E 04
660.0	3.4243E-01	5.1284E 02	1660.0	1.3703E 01	5.1618E 04
680.0	3.8586E-01	5.9540E 02	1680.0	1.4376E 01	5.4804E 04
700.0	4.3330E-01	6.8826E 02	1700.0	1.5073E 01	5.8144E 04
720.0	4.8498E-01	7.9236E 02	1720.0	1.5795E 01	6.1646E 04
740.0	5.4116E-01	9.0870E 02	1740.0	1.6542E 01	6.5314E 04
760.0	6.0207E-01	1.0383E 03	1760.0	1.7316E 01	6.9155E 04
780.0	6.6800E-01	1.1823E 03	1780.0	1.8117E 01	7.3175E 04
800.0	7.3919E-01	1.3419E 03	1800.0	1.8945E 01	7.7379E 04
820.0	8.1593E-01	1.5182E 03	1820.0	1.9801E 01	8.1775E 04
840.0	8.9849E-01	1.7126E 03	1840.0	2.0686E 01	8.6368E 04
860.0	9.8716E-01	1.9264E 03	1860.0	2.1600E 01	9.1165E 04
880.0	1.0822E 00	2.1611E 03	1880.0	2.2544E 01	9.6173E 04
900.0	1.1840E 00	2.4181E 03	1900.0	2.3518E 01	1.0140E 05
920.0	1.2928E 00	2.6990E 03	1920.0	2.4524E 01	1.0685E 05
940.0	1.4090E 00	3.0054E 03	1940.0	2.5562E 01	1.1253E 05
960.0	1.5328E 00	3.3390E 03	1960.0	2.6633E 01	1.1845E 05
980.0	1.6646E 00	3.7016E 03	1980.0	2.7737E 01	1.2462E 05
1000.0	1.8047E 00	4.0951E 03	2000.0	2.8875E 01	1.3104E 05
1020.0	1.9534E 00	4.5213E 03	2100.0	3.5097E 01	1.6725E 05
1040.0	2.1112E 00	4.9823E 03	2200.0	4.2275E 01	2.1105E 05
1060.0	2.2783E 00	5.4801E 03	2300.0	5.0502E 01	2.6357E 05
1080.0	2.4552E 00	6.0170E 03	2400.0	5.9874E 01	3.2608E 05
1100.0	2.6422E 00	6.5952E 03	2500.0	7.0495E 01	3.9991E 05
1120.0	2.8397E 00	7.2169E 03	2600.0	8.2469E 01	4.8655E 05
1140.0	3.0480E 00	7.8847E 03	2700.0	9.5907E 01	5.8760E 05
1160.0	3.2676E 00	8.6011E 03	2800.0	1.1092E 02	7.0478E 05
1180.0	3.4988E 00	9.3685E 03	2900.0	1.2764E 02	8.3995E 05
1200.0	3.7421E 00	1.0190E 04	3000.0	1.4618E 02	9.9510E 05
1220.0	3.9979E 00	1.1068E 04	3100.0	1.6666E 02	1.1724E 06
1240.0	4.2666E 00	1.2005E 04	3200.0	1.8923E 02	1.3741E 06
1260.0	4.5486E 00	1.3005E 04	3300.0	2.1402E 02	1.6026E 06
1280.0	4.8443E 00	1.4071E 04	3400.0	2.4116E 02	1.8606E 06
1300.0	5.1543E 00	1.5205E 04	3500.0	2.7081E 02	2.1508E 06
1320.0	5.4789E 00	1.6411E 04	3600.0	3.0311E 02	2.4761E 06
1340.0	5.8185E 00	1.7692E 04	3700.0	3.3822E 02	2.8397E 06
1360.0	6.1738E 00	1.9053E 04	3800.0	3.7630E 02	3.2447E 06
1380.0	6.5450E 00	2.0495E 04	3900.0	4.1750E 02	3.6948E 06
1400.0	6.9328E 00	2.2024E 04	4000.0	4.6199E 02	4.1934E 06
1420.0	7.3375E 00	2.3643E 04	4500.0	7.4002E 02	7.5566E 06
1440.0	7.7597E 00	2.5356E 04	5000.0	1.1279E 03	1.2797E 07

Table C2 (Continued)

$T(°K)$	$I(T)_{0-\infty}$	$B(T)_{max}$	$T(°K)$	$I(T)_{0-\infty}$	$B(T)_{max}$
5500.0	1.6514E 03	2.0610E 07	14000.0	6.9328E 04	2.2024E 09
6000.0	2.3388E 03	3.1843E 07	15000.0	9.1361E 04	3.1097E 09
6500.0	3.2214E 03	4.7515E 07	16000.0	1.1827E 05	4.2940E 09
7000.0	4.3330E 03	6.8826E 07	17000.0	1.5073E 05	5.8144E 09
7500.0	5.7101E 03	9.7178E 07	18000.0	1.8945E 05	7.7379E 09
8000.0	7.3919E 03	1.3419E 08	19000.0	2.3518E 05	1.0140E 10
8500.0	9.4204E 03	1.8170E 08	20000.0	2.8875E 05	1.3104E 10
9000.0	1.1840E 04	2.4181E 08	21000.0	3.5097E 05	1.6725E 10
9500.0	1.4699E 04	3.1687E 08	22000.0	4.2275E 05	2.1105E 10
10000.0	1.8047E 04	4.0951E 08	23000.0	5.0502E 05	2.6357E 10
11000.0	2.6422E 04	6.5952E 08	24000.0	5.9874E 05	3.2608E 10
12000.0	3.7421E 04	1.0190E 09	25000.0	7.0495E 05	3.9991E 10
13000.0	5.1543E 04	1.5205E 09			

Appendix D

FORTRAN IV Subprogram to Compute Either $I(T)_{\nu-\infty}$ OR $I(\nu/T)_{Norm}$

```
      FUNCTION PLNKIN(XNU, T, IOPT)
C   PLNKIN COMPUTES EITHER THE PLANCK INTEGRAL, I(T)NU TO INFINITY, OR
C   THE NORMALIZED INTEGRAL, I(NU/T)NORM, DEPENDING UPON THE OPTION
C   DECLARED BY THE USER.
C   IF IOPT = 1, I(T)NU TO INFINITY IS COMPUTED.
C   IF IOPT = 2, I(NU/T)NORM IS COMPUTED.
C   LIMITS ON Z ARE 1.17E-8 < Z < 90.14 WHERE
C         Z = C2*NU/T
      IF(IOPT.NE.1.AND.IOPT.NE.2) GO TO 500
      C2 =1.43879
      Y=XNU/T
      Z = C2*Y
      A2=EXP(Z)
      S=0.
      IF(Y.LT.7.4) GO TO 5
      X=2.
      GO TO 10
    5 IF(Y.LT.4.2) GO TO 6
      X=3.
      GO TO 10
    6 IF(Y.GT.0.45) GO TO 7
      X= AINT(-46.66*Y+33.)
      GO TO 10
    7 X= AINT(-2.13*Y +12.96)
   10 A1=EXP(-X*Z)
   15 A1=A1*A2
      X=X-1.
      ARG=X*Z
      S=S+A1*(6.+ARG*(6.+ARG*(3.+ARG)))/X**4
      IF(X.EQ.1.) GO TO (30,40),IOPT
      GO TO 15
C   COMPUTE PLANCK INTEGRAL(WATTS CM-2 STER-1)
   30 C1 =1.1909E-12
      PLNKIN = C1 * S *(T/C2)**4
   35 RETURN
C   COMPUTE NORMALIZED INTEGRAL
   40 PLNKIN = 0.15399*S
      GO TO 35
  500 WRITE(6,510) IOPT,XNU,T
  510 FORMAT('0OPTION SPECIFIED IN ROUTINE PLNKIN NE TO 1 OR 2'/' OPTION
     1 ='I5,' NU ='F6.3,' TEMP ='F9.2)
      STOP
      END
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